Regular local rings

Let A be a noetherian local ring, with maximal ideal m and residue field k. Then for each i, A/m^{i+1} as an A-module of finite length, $\ell_A(i)$. In fact for each i, m^i/m^{i+1} is a is a finite dimensional k-vector space, and $\ell_A(i) = \sum dim(m^j/m^{j+1}) : j \leq i$. It turns out that there is a polyonomial p_A with rational coefficients such that $p_A(i) = \ell_A(i)$ for i sufficiently large. Let d_A be the degree of p_A. The main theorem of dimension theory is the following:

Theorem 1 *Let A be a noetherian local ring. Then d_A is the Krull dimension of A, and this number is also the minimal length of a sequence $(a_1, \ldots a_d)$ of elements of m such that m is nilpotent moduloo the ideal generated by $(a_1, \ldots, a_d)$.*

Corollary 2 *If A is a noetherian local ring and a is an element of the maximal ideal of A, then*

$$dim(A/(a)) \geq dim(A) - 1,$$

with equality if a does not belong to any minimal prime of A (if and only if a does not belong to any of the minimal primes which are at the bottom of a chain of length the dimension of A).

Proof: Let $(a_1, \ldots, a_d)$ be sequence of elements of m lifting a minimal sequence in $m/(a)$ such that m is nilpotent modulo (a). Then d is the dimension of $A/(a)$. But now $(a_0, a_1, \ldots, a_d)$ is a sequence in m such that m is nilpotent. Hence the dimension of A is at most $d + 1$. □

Let A be a noetherian local ring and let $\mathrm{Gr}_m A := \oplus m_i/m^{i+1}$ which forms a graded k-algebra, generated in degree one by $V := m/m^2$. Then there is a natural surjective map

$$(*) \quad S^{\cdot}V \to \mathrm{Gr}_m A.$$

If d is the dimension of V then the dimension of S^iV is just the number of monomials of degree i in d variables, which is easily seen to be $\binom{d+i-1}{i}$ if $i \geq 0$. This is a polynomial of degree $d - 1$. It follows that the dimension of $\mathrm{Gr}_m A$ is at most $\binom{d+i-1}{i}$ and hence that $\ell_A(i)$ is bounded by a polynomial of degree d. It follows that the dimension of A is less than or equal to the dimension of the k-vector space V.

Definition 3 *Let A be a noetherian local ring with maximal ideal m and Krull dimension d. Then d is less than or equal to the dimension of m/m^2, and the ring is said to be regular if equality holds, and in this case (*) is an isomorphism and A is an integral domain.*

Proof: Note that $S^{\cdot}V$ is an integral domain. Suppose (*) is not surjective; choose a nonzer $f \in S^rV$ in the kernel K of (*). Then multiplication by f defines an injective map $S^{i-r}V \to K \cap S^i$, and it follows easily that the dimension of the quotient $\mathrm{Gr}^i A$ is less than or equal to $\binom{d+i-1}{i} - \binom{d+i-r-1}{i-r}$, a polynomial of degree $d - 1$, so A is not regular. On the other hand, if A is regular, (*) is an isomorphism, so $\mathrm{Gr}_m A$ is an integral domain. One concludes easily that A is also an integral domain. □

Proposition 4 *Let B be a regular local ring with maximal ideal m, and $A := B/I$, where I is a proper ideal of B. Then A is regular if and only if $I \cap m^2 = mI$, that is, if and only if the map $I/mI \to m/m^2$ is injective.*

Proof: Let $\overline{m} := m/m \cap I$ the maximal ideal of A, and let $\overline{V} := \overline{m}/\overline{m}^2$. Then we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & S^{\cdot}V & \longrightarrow & S^{\cdot}\overline{V} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Gr}_m I & \longrightarrow & \mathrm{Gr}_m B & \longrightarrow & \mathrm{Gr}_m A & \longrightarrow & 0 \end{array}$$

Since B is regular,the middle vertical arrow is bijective. Then A is regular if and only if the map on the right is injective, which is true if and only if the map on the left is surjective. Since K is generated in degree one and the map is an isomorphism in degree one, A is regular if and only if $\mathrm{Gr}_m I$ is generated in degree one. The degree i term of $\mathrm{Gr}_m I$ is $I \cap m^i/I \cap m^{i+1}$, so regularity of A is equivalent to saying that $I \cap m^i \subseteq Im^{i-1} + I \cap m^{i+1}$ for all i. If this is true, then we see by induction that $I \cap m^2 \subseteq Im + I \cap m^{i+1}$ for all i. The Artin–Rees lemma says that $I \cap m^{i+1} \subseteq mI$ for i sufficiently large, so we can conlude $I \cap m^2 \subseteq Im$, hence $I \cap m^2 = Im$. Conversely, say $I \cap m^2 = Im$, and choose elements $(x_1, \ldots x_r)$ of I lifting a basis of $I \cap m^2$. Then the dimension of $\overline{m}/\overline{m}^2$ is $N - r$, where N is the dimension of B. However, since $(x_1, \ldots x_r)$ generates I/mI it follows from Nakayama that it also generates I, and then that the dimension of A is at least $N - r$. But then we have equality, hence A is regular.

□

Theorem 5 *Let k be an algebraically closed field and let X/k be a scheme of finite type, and let x be a closed point of X. Then the following conditions are equivalent:*

1. *The local ring $\mathcal{O}_{X,x}$ is regular.*
2. *There is an open neighborhood U of x which is smooth over k.*

Proof: We may assume without loss of generality that X is affine, say $\operatorname{Spec} A$, where A is the quotient of a polynomial ring B over k by an ideal I. Let $Z := \operatorname{Spec} B$ and let m be the maximal ideal of B corresponding to the point $x \in X \subseteq Z$. Since the local ring B_m is regular, Proposition 4 says that A_m is regular if and only if the map $I/mI \to m/m^2$ is injective. Since x is a k-rational point of Z, the differential induces an isomoprhism $m/m^2 \to \Omega_{Z/k}(x)$, and Corollary 3 now says that this injectivity is equivalent to (2). □

Corollary 6 *If X/k is of finite type over an algebraically closed field k, then the set of points of x such that $\mathcal{O}_{X,x}$ is regular is open.*

Proof: To prove this we need to check that any localization of a regular local ring is regular, which we cannot do here. □